JOURNAL OF ALGEBRA 75, 484–494 (1982)

BN-Pairs and Finite Groups with Parabolic-Type Subgroups

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Communicated by David M. Goldschmidt

Received April 18, 1981

1. INTRODUCTION

In this paper we are concerned with a generalization of the BN-pair concept.

DEFINITION. A *partial BN-pair* in a group G is a pair of subgroups B and N together with a set $S \subseteq N/(B \cap N)$ satisfying the following.

(1) $G = \langle B, N \rangle$

(2) $B \cap N \trianglelefteq N$

(3) The set S generates the group $N/(B \cap N)$, and for each $s \in S$, $s^2 = 1$, and the following two conditions are satisfied:

(a) $BsBsB \subseteq B \cup BsB$,

(b) $sBs \neq B$.

Note that although s is not an element of G, Bs and BsB, etc., are well defined by interpreting s as any element of N corresponding to s via the natural map $N \to N/(B \cap N)$.

We will say that a partial BN-pair is *complete* if it is actually a BN-pair (i.e., if $BnBsB \subseteq BnB \cup BnsB$ for any $n \in N$ and $s \in S$). For a subset $R \subseteq S$ we define N_R to be the inverse image in N of the group $\langle R \rangle$ generated by R in $N/(B \cap N)$, and $G_R = \langle B, N_R \rangle$. If $R = \{s\}$ or $\{s, t\}$ we will use s or s, t as subscripts in place of $\{s\}$ and $\{s, t\}$. A partial BN-pair will be called *pairwise complete* if for each pair $u, v \in S$ we have $BnBsB \subseteq BnB \cup BnsB$ for $n \in N_{u,v}$ and $s \in \{u, v\}$. Note that the pair B, N_R together with the set R is a partial BN-pair in G_R. Thus a partial BN-pair is pairwise complete if B, N_R is complete for each two-element subset $R \subseteq S$. For brevity, we will call a pairwise complete partial BN-pair simply a *pairwise BN-pair.* Following the theory of BN-pairs we will call the subgroups G_R and their conjugates in G

0021-8693/82/040484–11$02.00/0

parabolic subgroups of G. For a subset $R \subseteq S$, let $B_R = \bigcap_{n \in N_R} B^n$. We use the same subscript convention as above when $R = \{s\}$ or $\{s, t\}$. Now we can state our first theorem.

THEOREM A. *Let G be a group with subgroups B and N which, together with a set $S \subseteq N/(B \cap N)$, form a pairwise BN-pair. Let $H = B \cap N$. Then B and N form a (complete) BN-pair for G provided the following further condition is satisfied for each pair s, $t \in S$.*

(*) If X is an H-invariant subgroup of B covering B/B_s and B/B_t, then X covers $B/B_{s,t}$. (We say that X covers Y/Z if $Y = X \cdot Z$. Y need not contain Z as a normal subgroup.)

Note that if $B_{s,t} \trianglelefteq G_{s,t}$, then the validity of condition (*) is determined by the structure of the group $G_{s,t}/B_{s,t}$. In Section 3 we determine the finite rank-2 Lie-type groups which fail to satisfy (*). In particular, there are only finitely many of them.

We are interested in applying Theorem A to the theory of finite groups. Tits [4] has classified the finite groups with BN-pairs of rank $\geqslant 3$. Theorem A can be used to extend his work to a classification of finite groups which have a "Lie-type" parabolic subgroup structure in ranks 1 and 2. In this paper we will use the following meaning for the term "Lie-type" group.

DEFINITION. By a *Lie-type group of characteristic p* we mean a group L satisfying

(1) $|Z(L)|$ is prime to p, and

(2) $L/Z(L) \simeq L_1 \times L_2 \times \cdots \times L_r$, where each L_i is a Chevalley or twisted group defined over a field of characteristic p in the sense of Carter [1].

When L is a Lie-type group, the groups L_i in (2) have a standard BN-pair when considered as Chevalley groups or twisted groups over a field of characteristic p. This gives in a natural way a BN-pair for L, which we will call the *standard* BN-pair for L when considered as a Lie-type group of characteristic p. We define the *rank* of L to be the rank of this BN-pair (i.e., the cardinality of the set $S \subseteq N/(B \cap N)$). Also, we will say L has *type* $L_1 \times \cdots \times L_r$.

THEOREM B. *Let G be a finite group and p a prime dividing $|G|$. Suppose G has subgroups $P_1, \ldots, P_r$ satisfying*

(1) $\langle P_1, \ldots, P_r \rangle = G$,

(2) *$P_1 \cap \cdots \cap P_r$ contains a Sylow p-subgroup of each group $P_{ij} = \langle P_i, P_j \rangle$ for all i and j,*

(3) $O^{p'}(P_i/O_p(P_i)) \simeq G_i$, *where* G_i *is some* rank-1 *Lie-type group of characteristic p for all i, and*

(4) $O^{p'}(P_{ij}/O_p(P_{ij})) \simeq G_{ij}$, *where* G_{ij} *is a* rank-2 *Lie-type group of characteristic p for all i and j.*

Finally, assume that no G_i *is of type* $A_1(2)$, $A_1(3)$, ${}^2A_2(4)$, ${}^2B_2(2)$, *or* ${}^2G_2(3)$, *and that no* G_{ij} *is of type* $A_2(4)$.

Then the subgroup $G_0 = \langle O^{p'}(P_1),..., O^{p'}(P_r)\rangle$ *is normal in G and has a BN-pair of* rank *r.*

Remark. The condition that G_{ij} not be of type $A_2(4)$ can be replaced by the weaker condition "If G_{ij} is of type $A_2(4)$, then $X = P_{ij}/O_p(P_{ij})$ satisfies $X/Z(X)$ is isomorphic to $PGL_3(4)$."

The proof of Theorem A is an adaptation of an argument discovered by Goldschmidt (see [2, 3.1]). The argument is quite elementary and short. Given its simplicity, it is surprising that the rather complex condition (*) is necessary for the argument. The author does not know if condition (*) is necessary for the theorem.

Given Theorem A, the proof of Theorem B is very easy, and consists in showing (essentially) that if U is a Sylow p-subgroup of G in Theorem B, if $B = N_G(U)$, and if H is a complement to U in B, then B and $N = N_G(H)$ form a partial BN-pair in G. The main difficulty is in obtaining $B \cap N \trianglelefteq N$, since H as defined here might include some "field automorphisms" of G. To avoid this problem, H must be more carefully defined. Condition (4) of Theorem B then ensures that we have a pairwise BN-pair, and so the proof of Theorem B is finished by calculating that the allowed Lie-type groups G_{ij} must satisfy condition (*) of Theorem A.

2. Proof of Theorem A

In this section G is a group with subgroups B and N which, together with a set $S \subseteq N/(B \cap N)$, form a pairwise BN-pair in G. We also assume that condition (*) of Theorem A is satisfied. For convenience, we repeat it here.

(*) For each pair s, $t \in S$, if X is an H-invariant subgroup of B which covers both B/B_s and B/B_t, then X also covers $B/B_{s,t}$.

We wish to prove that B, N is a BN-pair in G. The argument is an adaptation of Goldschmidt's argument in [2], and in order to make the parallel with his proof more apparent, we make the following definition. Let $W = N/(B \cap N)$.

Definition. For $s \in S$, let

$$D(s) = \{w \in W \mid B = (B^w \cap B) \cdot (B^s \cap B)\}.$$

LEMMA 2.1. *Assume that for each* $s \in S$, $W = D(s) \cup D(s) \cdot s$. *Then* B, N *is a BN-pair.*

Proof. Let $n \in N$, and let w be the image of n in W. Then $w \in D(s)$ or $w \in D(s) \cdot s$. In the first case, $(B^w \cap B) \cdot (B^s \cap B) = B$. In particular, $B \subseteq B^w \cdot B^s = n^{-1}$ $BnsBs$. Hence $nBs \subseteq BnsB$, so $BnBsB \subseteq BnsB$. In fact, since $BnBsB$ is a union of double cosets, we actually get $BnBsB = BnsB$. In the second case, $ws \in D(s)$, so by the above argument, $BnsBsB = BnssB = BnB$. Hence $BnBsB = BnsBsBsB$. By 2a of the definition of a partial BN-pair. $BsBsB \subseteq B \cup BsB$. Hence $BnBsB \subseteq Bns(B \cup BsB) = BnsB \cup BnsBsB = BnsB \cup BnB$. This proves the lemma.

The proof of Theorem A is accomplished by showing that an element w of W must lie in $D(s) \cup D(s) \cdot s$. This is done by induction on $l(w)$, the length of the shortest expression of w as a word in the elements of S. We need two other notions before beginning the argument. First, for $s, t \in s$, and $n \in \langle s, t \rangle$, let $l_{st}(n)$ denote the length of the shortest expression of n as a word in s and t. Second, let $w \in W$ and $s, t \in S$. By an s, t-decomposition of w, we mean an expression $w = w_1 w_0$, where $w_0 \in \langle s, t \rangle$, such that $l(w) = l(w_1) + l_{st}(w_0)$.

LEMMA 2.2. *For each* $w \in W$ *and* $s \in S$, *we have*

$$l(ws) \geqslant l(w) \Rightarrow w \in D(s).$$

Proof. This is a well-known fact when B, N is a BN-pair. A proof can be found in Richen [3, Lemma 2.5]. Since B, N_{st} is a BN-pair, we may assume that if $w \in \langle s, t \rangle$ for some t, then the conclusion of our lemma holds with l_{st} in place of l.

Now suppose the lemma is false, and let $w \in W$ be chosen so that $l(ws) \geqslant l(w)$ and $w \notin D(s)$ with $l(w)$ as small as possible. Let $t \in W$ be such that $w = w't$ with $l(w') < l(w)$. A moment's thought will reveal that this is an s, t-decomposition of w. In particular, non-trivial s, t-decompositions of w exist. Let $w = w_1 w_0$ be an s, t-decomposition with $l(w_1)$ as small as possible. Then since $l(w) = l(w_1) + l_{st}(w_0)$, and $l(ws) \geqslant l(w)$, another moment's thought will yield that $l_{st}(w_0 s) \geqslant l_{st}(w_0)$. Hence by the remark at the beginning of the proof, we have $w_0 \in D(s)$. So

$$B = (B^{w_0} \cap B) \cdot B_s. \tag{1}$$

Next, by our choice of $w = w_1 w_0$, we must have $l(w_1 s) \geqslant l(w_1) \leqslant l(w_1 t)$, since otherwise we could shorten w_1 in the st-decomposition of w. By minimality of w we conclude

$$B = (B^{w_1} \cap B) \cdot B_s = (B^{w_1} \cap B) \cdot B_t. \tag{2}$$

Let $X = B^{w_1} \cap B$. Since $H \subseteq \bigcap_{n \in N} B^n \subseteq X$, X is H-invariant. By (2) and assumption (*) of Theorem A, we conclude that X covers B/B_{st}. Using this information we can show $w \in D(s)$:

$$\begin{aligned}
(B^w \cap B) \cdot B_s &= (B^{w_1 w_0} \cap B) \cdot B_s = (B^{w_1 w_0} \cap B) \cdot B_{st} \cdot B_s \quad (\text{since } B_{st} \subseteq B_s)\\
&= (B^{w_1 w_0} \cdot B_{st} \cap B) \cdot B_s\\
&= (((B^{w_1}) \cdot B_{st})^{w_0} \cap B) \cdot B_s \quad (\text{since } B_{st}^{w_0} = B_{st})\\
&\supseteq ((X \cdot B_{st})^{w_0} \cap B) \cdot B_s\\
&= (B^{w_0} \cap B) \cdot B_s\\
&= B \qquad (\text{by } (1)).
\end{aligned}$$

This proves the lemma.

THEOREM A. *B, N is a BN-pair in G.*

Proof. By Lemma 2.1 we only need show $W = D(s) \cup D(s)s$. Let $w \in W$. If $l(ws) \geqslant l(w)$ then $w \in D(s)$ by Lemma 2.2. If $l(ws) < l(w)$ then $l(wss) > l(ws)$, so $ws \in D(s)$ and $w \in D(s)s$.

3. PROPERTIES OF THE LIE-TYPE GROUPS OF LOW RANK

In this section we sketch some calculations concerning the Lie-groups of ranks 1 and 2. In particular, we will determine all counterexamples to the following two statements about a Lie-type group of characteristic p.

(*) Suppose L has rank 2. Let B, N be the standard BN-pair of L. Let U be the Sylow p-subgroup of B. Let P_1 and P_2 be the two rank-1 parabolic subgroups containing B and let $U_i = O_p(P_i)$ $(i = 1, 2)$. Finally, let H be a complement to U in B. Then no proper H-invariant subgroup of U covers both U/U_1 and U/U_2.

(**) Suppose L has rank 1, U is a Sylow p-subgroup of L, $B = N_L(U)$, H is a complement to U in B, and $N = N_L(H)$. Then B, N is the standard BN-pair for L.

We remark that (*) is equivalent to condition (*) of Theorem A, and that (**) is needed for Theorem B in obtaining a partial BN-pair. We begin by discussing (**).

LEMMA 3.1. *Statement (**) holds if and only if $C_U(H) = 1$.*

Proof. Let N_0 be the monomial group of L. (See Carter [1].) We may assume that $N_0 \leqslant N$ and that B, N_0 is the standard BN-pair for L. Thus (**) is equivalent to $N = N_0$.

Now by Theorem 8.4.3, Proposition 13.5.3 of Carter [1], there is a canonical form for elements of L. If $g \in L$, then g has a unique expression

$$g = unv,$$

where $u \in U$, $n \in N_0$, and $v \in U_n^-$. Now $g \in N$ if and only if for each $h \in H$, there is $h' \in H$ such that $g = hunvh'$. In fact, $h' = (h^{-1})^g$. Now U and U_n^- are H-invariant, so that letting $u' = u^{h^{-1}}$ and $v' = v^{h'}$, we get $g = u'n'v'$, where $n' = hnh' \in N_0$, $u' \in U$, and $v' \in U_n^- = U_{n'}^-$. By uniqueness, we get $u' = u$ and $v' = v$. Since h was arbitrary, we get u and $v \in C_U(H)$. Thus $g \in N$ forces $g \in N_0$ iff $C_U(H) = 1$. This proves the lemma.

In Carter's proof of the simplicity of the Chevalley and Twisted groups (Theorems 11.1.2 and 14.4.1 of [1]) he shows that $C_U(H) = 1$ in all the rank 1 Chevalley and Twisted groups except $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$, ${}^2B_2(2)$, and ${}^2G_2(3)$. These are the Chevalley and Twisted groups of rank 1 which happen to not be simple.

Proposition 3.2. *Let L be a Lie-type group of* rank 1 *and characteristic p such that $L/Z(L)$ is simple. Then if $U \in \mathrm{Syl}_p(L)$, $B = N_L(U)$, H is a complement to U in B, and $N = N_L(H)$, then B, N is the standard BN-pair for L.*

Next we focus on condition (*).

Proposition 3.3. *Let L be a rank* 2 *Lie-type group. Then L satisfies* (*) *iff the following two conditions are satisfied*:

(a) $U_1 \cap U_2 \leqslant \Phi(U)$.

(b) *If* $U^i = U/U_i$ *and* $V^i = U^i/\Phi(U^i)$ $(i = 1, 2)$, *then, up to isomorphism,* V^1 *and* V^2 *have no common H-submodules* (i.e., $\mathrm{End}_H(V^1, V^2) = 0$).

Proof. Suppose (a) does not hold. Then there is an H-invariant complement $\bar{X}$ to $(U_1 \cap U_2)\,\Phi(U)/\Phi(U)$ in $U/\Phi(U)$. Let X be the inverse image of $\bar{X}$ in U. Then $X \neq U$, X is H-invariant, and $U_i \cdot X = U$ for $i = 1, 2$. So (*) does not hold.

Suppose (a) does hold. Then since $U_1 \cdot U_2 = U$ (this is true for any rank-2 BN-pair; see, for example, Lemma 2.5 in Richen [3]), we have $U/(U_1 \cap U_2) \simeq U^1 \times U^2$. Using (a) we have $\Phi(U/U_1 \cap U_2) = \Phi(U)/(U_1 \cap U_2)$. Thus $U/\Phi(U) \simeq V^1 \times V^2$.

Now if (a) holds but (b) does not, then using a "diagonal" submodule in $V^1 \times V^2$ we can construct an H-invariant subgroup $\bar{X}$ of $U/\Phi(U)$ covering both V^1 and V^2 and hence U^1 and U^2. The inverse image of $\bar{X}$ will then be a proper H-invariant subgroup of U covering U^1 and U^2 showing that (*) does not hold.

Conversely, suppose (a) and (b) do hold. Then the only H-submodule of $V^1 \times V^2$ projecting onto V^1 and V^2 is $V_1 \times V_2$ itself. Thus if X is an H-invariant subgroup of U covering U^1 and U^2, then passing to $U/\Phi(U)$ we see that X covers $U/\Phi(U)$. Hence X covers U.

PROPOSITION 3.4. *Suppose L_1 and L_2 are* rank-1 *Lie-type groups in characteristic p. Let $L = L_1 \times L_2$. If each L_i satisfies* (**), *then L satisfies* (*).

Proof. Let notation for L be chosen as in Proposition 3.3. Then (a) obviously holds. Also, $H = H_1 \times H_2$ where $H_i = H \cap L_i$ $(i = 1, 2)$. Also $U = U_1 \times U_2 \simeq U^1 \times U^2$, and by Lemma 3.1, $C_{U^i}(H_i) = 1$ $(i = 1, 2)$. Thus $C_{V^i}(H_i) = 1$ $(i = 1, 2)$. But H_j centralizes V^i for $j \neq i$, so an easy argument shows that (b) in Proposition 3.3 holds as well.

PROPOSITION 3.5. *Let L be a* rank-2 *Lie-type group for which $L/Z(L)$ is not isomorphic to one of the following types of groups: $L_1 \times L_2$, where L_1 and L_2 are* rank-1 *Chevalley or Twisted groups, $G_2(2)$, $G_2(3)$, or ${}^2F_4(2)$. Then condition* (a) *of Proposition* 3.3 *is satisfied.*

Proof. The proof of this proposition is a straightforward calculation. However, it is very long, and we do not wish to include it here.

PROPOSITION 3.6. *Let L be a* rank 2 *Lie-type group of characteristic p for which $L/Z(L)$ is not isomorphic to one of the following types of groups: $L_1 \times L_2$, where L_1 and L_2 are* rank-1 *Chevalley or Twisted groups, $A_2(3)$, $A_2(4)$, $B_2(2)$, $B_2(4)$, $G_2(2)$, ${}^2A_3(2^2)$, ${}^2A_3(3^2)$, ${}^3D_4(2^3)$, or ${}^2F_4(2)$. Then letting H be as in* (*), *H is non-cyclic.*

Proof. Again, this is a simple, but long, calculation, which we omit. It simply uses Carter's characterization of H in terms of K-characters [1, Theorems 7.1.1, 13.7.2, 13.7.4.]).

PROPOSITION 3.7. *Let L be a* rank-1 *Lie-type group which satisfies* (**). *Let U and H be as in* (**). *Then $U/\Phi(U)$ is a homogeneous H-module* (i.e., *it decomposes as a direct sum of isomorphic copies of an irreducible H-module).*

Proof. A simple calculation shows that $U/\Phi(U)$ is always isomorphic to a root group $X_r = \{x_r(t) \mid t \in GF(p^n)\}$ for some n, and some root r, except in the strange case of ${}^2G_2(3)$, which does not satisfy (**). Since $h = h(\chi)$ acts on X_r via the relation $x_r(t)^h = x_r(\chi(r) \cdot t)$, we see that when L satisfies (**) H acts homogeneously on X_r and hence on $U/\Phi(U)$.

THEOREM 3.8. *Let L be a* rank-2 *Lie-type group. Let P_1 and P_2 be its* rank-1 *parabolic subgroups containing a fixed B. Let $L_i = O^{p'}(P_i/O_p(P_i))$ $(i = 1, 2)$. Then if each L_i satisfies (**), L satisfies (*), except when L is of type $A_2(4)$.*

Proof. By Proposition 3.4 we may assume $L/Z(L)$ is not of type $L_1 \times L_2$. By Propositions 3.5 and 3.3 we need only show that (b) of Proposition 3.3 holds. Now letting notation be as in that proposition, we get from Proposition 3.7 that V^1 and V^2 are homogeneous H-modules. Now H acts faithfully on $V^1 \times V^2$, for if $h \in H$ centralized V^1 and V^2, then h would centralize U, which would say that $h = h(\chi)$, where χ is trivial on all roots. So H is faithful on $V^1 \times V^2$. Hence as H is non-cyclic by Proposition 3.5, (b) follows at once.

PROPOSITION 3.9. *If $L/Z(L)$ is isomorphic to a subgroup of $PGL_3(4)$ properly containing $PSL_3(4)$, then L satisfies (*).*

4. PROOF OF THEOREM B

In this section we will prove Theorem B. The proof is quite simple except for one technical difficulty. The problem is that the group G of Theorem B could itself be much larger than a Lie-type group—for example, a Lie-type group extended by some field automorphisms. Thus in constructing a pairwise BN-pair from the given hypotheses, one does not want to take B to be the normalizer in G of the Sylow p-subgroup U of $P_1 \cap \cdots \cap P_r$, since this will include the field automorphisms in B. Instead, we must define $B_i = N_{\hat{P}_i}(U)$, where $\hat{P}_i = O^{p'}(P_i)$, and then $B = \langle B_1, ..., B_r \rangle$. Our first result, which we will use in proving that the partial BN-pair we construct is actually a pairwise BN-pair, shows that if we have the pre-knowledge that G is a Lie-type group, then this construction gives us the B we want.

LEMMA 4.1. *Let G be a Lie-type group of characteristic p. Let B, N be the standard B, N-pair for G, let U be the Sylow p-subgroup of B, and let $P_1, ..., P_r$ be the* rank-1 *parabolic subgroups containing B. Finally let $\hat{P}_i = O^{p'}(P_i)$ and $B_i = N_{\hat{P}_i}(U)$ $(i = 1, ..., r)$. If $G = \langle P_1, ..., P_r \rangle$, then $B = \langle B_1, ..., B_r \rangle$.*

Proof. Let $B_0 = \langle B_1, ..., B_r \rangle$, $H = B \cap N$, $H_0 = B_0 \cap N$, $H_i = H \cap P_i$, and $N_i = N \cap P_i$. Then $H_0 = H_1 \cdot \cdots \cdot H_r$, and, if $N_0 = \langle N_1, ..., N_r \rangle$, we have $N = N_0 \cdot H$. We will show that B_0, N_0 is a BN-pair for G. Thus we will be able to conclude $B = B_0$, for B will be a rank-0 parabolic subgroup relative to B_0 and N_0.

First, since $[H, N_i] \leqslant H_i$, we get $H_0 \trianglelefteq N_0$. Thus $B_0 \cap N_0 = H_0 \trianglelefteq N_0$.

Now each N_i passes to a group of order 2 in N_0/H_0, so if we let s_i be the non-identity element of $N_i \cdot H_0/H_0$, we see that $S = \{s_1,..., s_r\}$ is a set of involutions generating N_0/H_0.

It is clear that each pair B_i, N_i is a BN-pair. Thus we see that B_0 and N_0, together with the set S, form a partial BN-pair. Furthermore, since G is a Lie-type group, we have, for $n \in N_0$ and $s \in S$,

$$l(ns) > l(n) \Rightarrow U^n \cdot U^s \supseteq U.$$

Thus we get

$$l(ns) > l(n) \Rightarrow B_0^n \cdot B_0^s \supseteq B_0.$$

Now using a standard argument (see the proof of Lemma 2.1) we can prove that B_0, N_0 is a BN-pair. This proves the lemma.

We now begin the proof of Theorem B. As we carry out the proof we will assume only as much of the hypothesis of Theorem B as is needed. We begin by making the following assumptions, which will be kept for the remainder of this section.

(1) G is a finite group, p is a prime dividing $|G|$, and $P_1,..., P_r$ are subgroups of G.

(2) $\langle P_1,..., P_r\rangle = G$.

(3) $P_1 \cap \cdots \cap P_r$ contains a Sylow p-subgroup of each $P_{ij} = \langle P_i, P_j\rangle$.

(4) $O^{p'}(P_i/O_p(P_i)) = G_i$ is a rank-1 Lie-type group of characteristic p.

(5) $O^{p'}(P_{ij}/O_p(P_{ij})) = G_{ij}$ is a rank-2 Lie-type group of characteristic p.

We also fix the following notation.

$\hat{P}_i = O^{p'}(P_i) \quad (i = 1,..., r).$

$\hat{P}_{ij} = O^{p'}(P_{ij}) \quad (\text{all } i, j).$

U is a Sylow p-subgroup of $P_1 \cap \cdots \cap P_r$.

$B_i = N_{\hat{P}_i}(U) \quad (i = 1,..., r).$

$B = \langle B_1,..., B_r\rangle.$

LEMMA 4.2. *U is a Sylow p-subgroup of B, and B normalizes each P_i.*

Proof. Since $U \trianglelefteq B$, we can, for $X \leqslant B$, let $\bar{X} = XU/U$. Thus $\bar{B} = \langle \bar{B}_1,..., \bar{B}_r\rangle$. Now since $U \in \mathrm{Syl}_p(P_{ij})$ and $\hat{P}_{ij}/O_p(\hat{P}_{ij}) = G_{ij}$ is a Lie-type group of characteristic p, we get that $\bar{B}_i$ and $\bar{B}_j$ normalize each other and $\langle \bar{B}_i, \bar{B}_j\rangle$ is a p'-group for each pair i, j. Thus $\bar{B}$ is a p'-group as well. Also, B_i normalizes P_j, so B does as well.

For the remainder of this section we fix the following further notation.

H is some complement to U in B.

$H_i = H \cap \hat{P}_i \quad (i = 1,..., r)$.

$N_i = N_{\hat{P}_i}(H), \quad (i = 1,..., r)$.

$N = \langle N_1,..., N_r \rangle$.

Furthermore, for the remainder of this section we will assume that

(6) Condition (**) of Section 3 holds with each G_i in place of L.

LEMMA 4.3. *For each i, N_i covers a monomial subgroup of G_i, so that B_i, N_i is a BN-pair for $\hat{P}_i$.*

Proof. First note that $\hat{P}_i/O_p(\hat{P}_i) = G_i$. For $X \leqslant \hat{P}_i \cdot H$, let $\bar{X}$ denote the image of X in $\hat{P}_i \cdot H/O_p(\hat{P}_i)$. We wish to show that $\bar{N}_i$ is a monomial subgroup of G_i. By (6) it suffices to show $\bar{N}_i = N_{G_i}(\bar{H}_i)$. Since $N_i = N\hat{P}_i(H) \leqslant N_{\hat{P}_i}(H_i)$, we get $\bar{N}_i \leqslant N_{G_i}(\bar{H}_i)$. As $N_{G_i}(\bar{H}_i)$ is normalized by $\bar{H}$ and $N_{G_i}(\bar{H}_i)/\bar{H}_i$ has order 2, $[N_{G_i}(\bar{H}_i), \bar{H}] \leqslant \bar{H}_i \leqslant H$. Thus $N_{G_i}(\bar{H}_i) \leqslant N_{G_i}(\bar{H}) = \bar{N}_i$.

LEMMA 4.4. $\langle B, N \rangle = \langle \hat{P}_1,..., \hat{P}_r \rangle \trianglelefteq G$.

Proof. Since B_i and N_i are in $\hat{P}_i$, $B = \langle B_1,..., B_r \rangle$, and $N = \langle N_1,..., N_r \rangle$, we get $\langle B, N \rangle \leqslant \langle \hat{P}_1,..., \hat{P}_r \rangle$. Since N_i covers a monomial group of G_i, we also get $\hat{P}_i = \langle B_i, N_i \rangle$. So $\langle B, N \rangle = \langle \hat{P}_1,..., \hat{P}_r \rangle$.

Now $\hat{P}_{ij} = \langle \hat{P}_i, \hat{P}_j \rangle$. Hence also, for fixed i, we get $\langle B, N \rangle = \langle \hat{P}_{ij} \mid j \neq i \rangle$. Since $P_i \leqslant P_{ij}$, we see that P_i normalizes $\hat{P}_{ij}$ for all $j \neq i$. Hence P_i normalizes $\langle B, N \rangle$. Since i was arbitrary and $G = \langle P_1,..., P_r \rangle$, we get that $\langle B, N \rangle \trianglelefteq G$.

LEMMA 4.5. $B \cap N \trianglelefteq N$.

Proof. As $N \leqslant N_G(H)$, $B \cap N \leqslant N_B(H)$. Now $N_B(H) = C_U(H) \cdot H$. As $U \leqslant \hat{P}_i$ for all i, $C_U(H) \leqslant N$, and so $N_B(H) \leqslant B \cap N$. Thus $B \cap N = N_B(H) = C_U(H) \cdot H$. Now $C_U(H) \leqslant C_U(H_i) \leqslant O_p(\hat{P}_i)$ by (6) and Lemma 3.1. Thus $C_U(H) = C_{O_p(\hat{P}_i)}(H)$ is normalized by N_i. As i was arbitrary, N normalizes $C_U(H)$. Thus $B \cap N \trianglelefteq N$.

LEMMA 4.6. *$N_i \cdot (B \cap N)/(B \cap N)$ has order 2 for each i.*

Proof. This follows immediately from (6) and Lemma 4.3.

We now introduce further notation which we fix for the remainder of this section.

s_i is the non-identity element of $N_i \cdot (B \cap N)/(B \cap N)$ $(i = 1,..., r)$,
$s = \{s_1,..., s_r\}$.

PROPOSITION 4.7. *B and N, together with the set* $S \subseteq N/(B \cap N)$, *form a pairwise BN-pair for the normal subgroup* $G_0 = \langle B, N \rangle$ *of G.*

Proof. B_i and N_i form a rank-1 BN-pair in $\hat{P}_i$, since this is clearly true after passing to the homomorphic image G_i. Since $B = B_i \cdot H$ and $N_i \cdot (B \cap N) = N_i \cdot H$, the same is true of B and $N_i \cdot (B \cap N)$. Hence B and N form a *partial* BN-pair. It remains to show that we also have a pairwise BN-pair.

Let $N_{ij} = \langle N_i, N_j \rangle$. Let $B_{ij} = \langle B_i, B_j \rangle$. In order to prove that B_{ij}, N_{ij} forms a BN-pair we need to know that B_{ij} is a Borel subgroup as it should be. This, however, is the content of Lemma 4.1, and thus B_{ij}, N_{ij} is a rank-2 BN-pair. Hence if $R = \{s_i, s_j\}$, we get $B = B_{ij} \cdot H$ and $N_R = N_{ij} \cdot H$ form a rank-2 BN-pair for all i and j. This proves the proposition.

Combining Proposition 4.7 with Theorem A, we get the following theorem.

THEOREM 4.10. *With all notation as developed so far and the additional assumption that each* G_{ij} *satisfies condition* (*) *of Section* 3, *we have that B and N form a BN-pair for* G_0.

And finally, combining Proposition 3.2 and Theorems 3.8 and 4.10, we obtain Theorem B.

ACKNOWLEDGMENTS

The inspiration for this work came from reading Patrick McBride's thesis, where a similar situation is considered. Also we wish to thank Professors Andrew Chermak, David Goldschmidt, and Gary Seitz for many helpful comments.

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